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TWO MATRIX METHODS FOR CALCULATING FORCING
FUNCTIONS FROM KNOWN RESPONSES

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SUMMARY

Two matrix methods for calculating responses of linear systems for given forcing functions and indicial responses are analyzed in order to find a method suitable for solving the inverse problem of calculating forcing functions from known responses. The first method consists of a numerical evaluation of Duhamel's integral, the second of an approximation of the forcing function by means of straight-line or parabolic-arc segments. The second method is more generally adaptable to the solution of the inverse problem than the first; when the first method can be used, however, it is usually less time consuming.

The results of both methods are compared with exact solutions for both the direct and the inverse problems. Both methods are found to be convenient to use and sufficiently accurate for many practical purposes. They may find application in work on gust loads, maneuvering loads, impact loads, in electric problems, as well as many other dynamic and some static problems.

INTRODUCTION

It is often of interest to calculate the response of a static or dynamic system to an arbitrary forcing function, such as the response of an airplane to an arbitrary gust or stick-force variation. Similarly, it is often of interest to calculate the forcing function that gives rise to a known response.

If the system is linear so that solutions may be superposed, the direct problem of calculating the response to an arbitrary forcing function may be solved by superposition with the use of an indicial response (the response to a unit jump function, see fig. 1). The integral form of this superposition is known as Duhamel's integral. When the integral cannot be evaluated in closed form either numerical or graphical methods (reference 1) may be used.

For the purpose of solving the inverse problem of calculating forcing functions from known responses, Duhamel's integral may be considered to be an integro-differential equation. Because solutions to this equation cannot always be obtained in closed form and graphical methods cannot conveniently be adapted to this problem, numerical methods must be resorted to.

In this paper two numerical methods are presented for solving the inverse problem. One method consists of a numerical evaluation of Duhamel's integral in a manner similar to that of Simpson's rule. The other method consists of an approximation of the forcing function by means of straight-line or parabolic-arc segments. Both methods are expressed in matrix form and are based on matrices that are obtained by inverting matrices which constitute the solution to the direct problem. Consequently, the solution of the direct problem by the two matrix methods is discussed first and the adaptation of these methods to the solution of the inverse problem, later. The mechanics of computing solutions to specific problems are discussed in some detail in the corresponding sections of this paper concerned with the analysis of the direct and inverse problems. Methods of evaluating the responses to unit gradient and unit parabolic forcing functions (see fig. 1) required in the second method of this paper are described in the appendix.

The adaptation of these and similar numerical methods to the solution of the other inverse problem, that of calculating indicial responses from known forcing functions and responses, is discussed briefly.

SYMBOLS

s	independent variable, usually time
A	indicial response
G	arbitrary forcing function
R	response to arbitrary forcing function
B	response to unit gradient function
C	response to unit parabolic function
f	integrand of Duhamel's integral
σ	variable of integration corresponding to s
ζ, ξ	independent variables equivalent to s

$[D]$	differentiating matrix
$[Q]$	matrix which performs the function of Duhamel's integral
M, N	constants of linear and parabolic term in the parabolic-arc approximation
n	n th point on any curve or number of last point of interest
a	coefficient in Q or Q_1 matrix or coefficient of original matrix
b	coefficient in inverse of Q or Q_1 matrix or coefficient in auxiliary matrix
c, d, e, f	coefficients in Q_2 matrix
$\underline{g}$	submatrix of Q_2 matrix
$\underline{h}$	submatrix of inverse of Q_2 matrix

Matrix notation:

$[]$	square matrix
$\{\}$	column matrix

The subscripts on the A , B , C , G , and R functions or their derivatives denote the point at which the function (or derivative) is taken; thus, for instance, $G'_3 \equiv G'(s)_{s=3\Delta s}$.

A prime mark indicates differentiation with respect to s or σ . An underscore indicates a submatrix.

ANALYSIS OF THE DIRECT PROBLEM

Numerical Evaluation of Duhamel's Integral

The response $R(s)$ of a linear static or dynamic system to an arbitrary forcing function $G(s)$ can be obtained from the indicial response of the system $A(s)$ by superposition. If the forcing

function is approximated by a staircase function (fig. 2), the response at different values of the independent variable s will be:

$$\left. \begin{aligned} R(0) &\equiv R_0 \approx A_0 G_0 \\ R(s_1) &\equiv R_1 \approx A_1 G_0 + A_0 (G_1 - G_0) \\ R(s_2) &\equiv R_2 \approx A_2 G_0 + A_1 (G_1 - G_0) + A_0 (G_2 - G_1) \\ R(s_3) &\equiv R_3 \approx A_3 G_0 + A_2 (G_1 - G_0) + A_1 (G_2 - G_1) + A_0 (G_3 - G_2) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \right\} \quad (1)$$

As the interval Δs approaches zero these discrete values of R approach a function $R(s)$ given by Duhamel's integral:

$$R(s) = A(s)G_0 + \int_0^s A(s - \sigma)G'(\sigma)d\sigma \quad (2)$$

where σ is the variable of integration corresponding to s .

For cases in which G_0 is not zero the integral in equation (2) can be evaluated as indicated in the following analysis; to this integral the function $A(s)G_0$ may then be added. When G_0 is zero, equation (2) reduces to the simpler form:

$$R(s) = \int_0^s A(s - \sigma)G'(\sigma)d\sigma \quad (2a)$$

This expression is evaluated numerically in two steps: an expression for the derivative of the forcing function $G(\sigma)$ is first determined and then the integration of the function $A(s - \sigma)G'(\sigma)$ is performed by Simpson's rule.

An approximate value of the derivative of G at point s_n may be obtained from the relation

$$G'_n \approx \frac{G_{n+\frac{1}{2}} - G_{n-\frac{1}{2}}}{\Delta s} \quad (3)$$

which in effect implies that the actual curve for G is replaced by a parabolic segment between the points $s_{n-\frac{1}{2}}$ and $s_{n+\frac{1}{2}}$. If, in addition, the slope at $s = 0$ is estimated as

$$G'_0 \approx \frac{G_{1/2}}{\Delta s/2} \quad (3a)$$

a differentiating matrix for calculating $G'(s)$ can be set up from equations (3) and (3a):

$$\begin{Bmatrix} G'_0 \\ G'_1 \\ G'_2 \\ G'_3 \\ . \end{Bmatrix} \approx \frac{1}{\Delta s} \begin{bmatrix} 2 & 0 & 0 & 0 & . & . \\ -1 & 1 & 0 & 0 & . & . \\ 0 & -1 & 1 & 0 & . & . \\ 0 & 0 & -1 & 1 & . & . \\ . & . & . & . & . & . \end{bmatrix} \begin{Bmatrix} G_{1/2} \\ G_{3/2} \\ G_{5/2} \\ G_{7/2} \\ . \end{Bmatrix} \quad (4)$$

or

$$\{G'\} \approx \frac{1}{\Delta s} [D] \{G\} \quad (4a)$$

The integration of the function $A(s - \sigma)G'(\sigma) \equiv f(\sigma)$ indicated in equation (2a) may be performed by Simpson's rule. For even values of n ,

$$\begin{aligned} R_n &= \int_0^{\sigma_n} f(\sigma) d\sigma \\ &\approx \frac{\Delta s}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 4f_{n-1} + f_n) \end{aligned} \quad (5)$$

For odd intervals the integration over the last segment s_{n-1} to s_n can be performed by passing a parabola through the three points s_{n-2} , s_{n-1} , and s_n and integrating the parabola between s_{n-1} and s_n . This procedure yields a set of coefficients ($-1/12$, $2/3$, and $5/12$) which is used in the same manner as the Simpson integrating factors ($1/3$, $4/3$, and $1/3$) which would be obtained upon integrating from s_{n-2} to s_n .

Thus for odd values of n ,

$$\begin{aligned}
 R_n &= \int_0^{\sigma_n} f(\sigma) d\sigma \\
 &\approx \frac{\Delta s}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 4f_{n-4} + 2f_{n-3}) \\
 &\quad + \Delta s \left(\frac{5}{4} f_{n-2} + f_{n-1} + \frac{5}{12} f_n \right)
 \end{aligned} \tag{5a}$$

Since there are only two values of the function f that can be used to calculate R_1 , the trapezoidal rule is used in the initial interval instead of Simpson's rule, so that

$$R_1 \approx \frac{\Delta s}{2} (f_1 + f_0) \tag{5b}$$

Equations (5), (5a), and (5b) can be combined by writing them in matrix form:

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ \vdots \end{Bmatrix} \approx \Delta s \begin{bmatrix} \frac{1}{2}A_1 & \frac{1}{2}A_0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3}A_2 & \frac{4}{3}A_1 & \frac{1}{3}A_0 & 0 & 0 & 0 & \dots \\ \frac{1}{3}A_3 & \frac{5}{4}A_2 & A_1 & \frac{5}{12}A_0 & 0 & 0 & \dots \\ \frac{1}{3}A_4 & \frac{4}{3}A_3 & \frac{2}{3}A_2 & \frac{4}{3}A_1 & \frac{1}{3}A_0 & 0 & \dots \\ \frac{1}{3}A_5 & \frac{4}{3}A_4 & \frac{2}{3}A_3 & \frac{5}{4}A_2 & A_1 & \frac{5}{12}A_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} G'_0 \\ G'_1 \\ G'_2 \\ G'_3 \\ G'_4 \\ \vdots \end{Bmatrix} \tag{6}$$

or

$$\{R\} \approx \Delta s [A] \{G'\} \tag{6a}$$

Then, substituting the expression for $\{G'\}$ from equation (4a) into equation (6a) gives

$$\{R\} \approx [A] [D] \{G\} \tag{6b}$$

$$= [Q] \{G\} \tag{6c}$$

where $[Q]$ is a combined matrix that performs the operations indicated by Duhamel's integral numerically.

The steps required in computing responses by this method can be summarized as follows:

1. The values of the forcing function are tabulated for the midpoint of each interval and those of the indicial function are tabulated for each end point.

2. The values of the forcing function are then multiplied by the D matrix (equations (4) and (4a)) to obtain the values of a matrix that consists of Δs times the values of G' (the derivative of the forcing function). If n is the last point of interest, the tabulation of the values of the forcing function should be extended to the value at the midpoint of the next interval $G_{n+\frac{1}{2}}$; the values of G' can then be calculated up to G'_n .

3. The values of the indicial response are multiplied by the factors $1/2$, $1/3$, $5/12$, $5/4$, and so forth and tabulated in the manner indicated by the A matrix of equation (6). The A matrix has one more column than it has rows. If n is the last point of interest, it will have $n+1$ columns, which correspond to the $n+1$ values of G' (starting with G'_0).

4. The A matrix is postmultiplied by the column $\Delta s \{G'\}$ obtained previously. The values calculated in this manner are the values of the response at the end points of the intervals, provided G_0 is zero. Only n values of R are calculated; R_0 is zero.

5. If G_0 is not zero, the response is obtained by adding the terms $G_0 A_1$, $G_0 A_2$, $G_0 A_3$, . . . to the values of R_1 , R_2 , R_3 , . . . calculated in the preceding step. The initial value of the response is then $G_0 A_0$.

Approximation of the Forcing Function

In the preceding section the response to an arbitrary forcing function was evaluated numerically by approximating the integrand $A(s - \sigma)G'(\sigma)$ of Duhamel's integral by parabolic segments. Another method of calculating such a response consists of approximating the forcing function either by straight-line segments or by parabolic-arc segments. The response is then calculated as the linear superposition of responses to the unit gradient and unit parabolic forcing functions shown in figure 1.

Straight-line approximation.— If the arbitrary forcing function $G(s)$ is approximated by straight-line segments as shown in figure 3, the values

of the response to that forcing function at different values of s can be determined from the response $B(s)$ to a unit gradient function (see fig. 4) as follows:

$$\left. \begin{aligned} R_0 &= 0 \\ R_1 &\approx B_1 G_1 \\ R_2 &\approx B_2 G_1 + B_1 (G_2 - G_1) \\ R_3 &\approx B_3 G_1 + B_2 (G_2 - G_1) + B_1 (G_3 - G_2) \\ &\vdots \end{aligned} \right\} \quad (7)$$

This relation can be expressed in matrix form as

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ \vdots \end{Bmatrix} \approx \begin{bmatrix} B_1 & 0 & 0 & 0 & 0 & \dots \\ B_2 & B_1 & 0 & 0 & 0 & \dots \\ B_3 & B_2 & B_1 & 0 & 0 & \dots \\ B_4 & B_3 & B_2 & B_1 & 0 & \dots \\ B_5 & B_4 & B_3 & B_2 & B_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ \vdots \end{Bmatrix} \quad (7a)$$

or

$$\{R\} \approx [B][D_1]\{G\} \quad (7b)$$

$$= [Q_1]\{G\} \quad (7c)$$

A method for calculating the response to the unit gradient function $B(s)$ in terms of the indicial response $A(s)$ is presented in the appendix. The steps required in computing responses by this method can then be summarized as follows:

1. The values of the function G are tabulated at the end points of the intervals from point 1 to point n , and those of the function A , at the midpoints as well as at the end points from point 0 to point n .

2. The values of the function B at the end points of the intervals (from points 1 to n) are calculated from equation (A6) of the appendix. The matrix of equation (A6) will have n rows and $2n + 1$ columns, since $2n + 1$ values of A are used (at the n end points, at the n midpoints, and at 0) and n values of B are calculated.

3. The values of the function B calculated in this manner are entered in a matrix as shown in equation (7a) and are postmultiplied by the D_1 matrix to yield the Q_1 matrix. Both the D_1 and the Q_1 matrices are square and of order n .

4. The response to the forcing function $G(s)$ is found by postmultiplying the Q_1 matrix by the G matrix as indicated in equation (7c), provided G_0 is zero.

5. If G_0 is not zero, it must be subtracted from every other value of G in tabulating the G matrix. The terms $G_0A_1, G_0A_2, \dots$ must then be added to the values of the response calculated in the preceding step for points 1, 2, $\dots$. The response at point 0 is then G_0A_0 .

Parabolic-arc approximation.—A closer approximation to the forcing function than the straight-line approximation of the preceding section may be had by fitting parabolic-arc segments to the function. In the first segment $0 \leq s \leq \Delta s$, for instance, the forcing function may be expressed approximately as

$$G(s) \approx M_1 \left(\frac{s}{\Delta s} \right)^2 + N_1 \left(\frac{s}{\Delta s} \right) \quad (8)$$

provided G_0 is zero. The constants M_1 and N_1 are selected in such a manner that the parabolic arc coincides with the true values of G at $s = 0$, $s = \frac{\Delta s}{2}$, and $s = \Delta s$; therefore,

$$\left. \begin{aligned} M_1 &= -4G_1/2 + 2G_1 \\ N_1 &= 4G_1/2 - G_1 \end{aligned} \right\} \quad (9)$$

similarly, in the second interval

$$G(s) - G_1 \approx M_2 \left(\frac{s - s_1}{\Delta s} \right)^2 + N_2 \left(\frac{s - s_1}{\Delta s} \right) \quad (8a)$$

where

$$\left. \begin{aligned} M_2 &= -4(G_3/2 - G_1) + 2(G_2 - G_1) \\ N_2 &= 4(G_3/2 - G_1) - (G_2 - G_1) \end{aligned} \right\} \quad (9a)$$

and in further intervals, for $s_{n-1} \leq s \leq s_n$,

$$G(s) - G_{n-1} \approx M_n \left(\frac{s - s_{n-1}}{\Delta s} \right)^2 + N_n \left(\frac{s - s_{n-1}}{\Delta s} \right) \quad (8b)$$

where

$$\left. \begin{aligned} M_n &= -4 \left(G_{n-\frac{1}{2}} - G_{n-1} \right) + 2 \left(G_n - G_{n-1} \right) \\ N_n &= 4 \left(G_{n-\frac{1}{2}} - G_{n-1} \right) - \left(G_n - G_{n-1} \right) \end{aligned} \right\} \quad (9b)$$

These relations may be written in matrix form as

$$\begin{Bmatrix} M_1 \\ N_1 \\ M_2 \\ N_2 \\ M_3 \\ N_3 \\ . \end{Bmatrix} = \begin{bmatrix} -4 & 2 & 0 & 0 & 0 & 0 & . & . & . \\ 4 & -1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & -4 & 2 & 0 & 0 & . & . & . \\ 0 & -3 & 4 & -1 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 2 & -4 & 2 & . & . & . \\ 0 & 0 & 0 & -3 & 4 & -1 & . & . & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix} \begin{Bmatrix} G_{1/2} \\ G_1 \\ G_{3/2} \\ G_2 \\ G_{5/2} \\ G_3 \\ . \end{Bmatrix} \quad (9c)$$

or

$$\{M\} = [D_2] \{G\} \quad (9d)$$

Once the M and N coefficients pertaining to the arbitrary forcing function $G(s)$ have been evaluated, the response to that forcing function at different values of s may be determined from the response $B(s)$ to a unit gradient function and the response $C(s)$ to a unit parabolic function as follows:

$$\left. \begin{aligned}
 R_{1/2} &\approx C_{1/2}M_1 + B_{1/2}N_1 \\
 R_1 &\approx C_1M_1 + B_1N_1 \\
 R_{3/2} &\approx C_{3/2}M_1 + B_{3/2}N_1 + C_{1/2}M_2 + B_{1/2}N_2 \\
 R_2 &\approx C_2M_1 + B_2N_1 + C_1M_2 + B_1N_2 \\
 R_{5/2} &\approx C_{5/2}M_1 + B_{5/2}N_1 + C_{3/2}M_2 + B_{3/2}N_2 + C_{1/2}M_3 + B_{1/2}N_3
 \end{aligned} \right\} \quad (10)$$

or

$$\begin{Bmatrix} R_{1/2} \\ R_1 \\ R_{3/2} \\ R_2 \\ R_{5/2} \\ . \end{Bmatrix} \approx \begin{bmatrix} C_{1/2} & B_{1/2} & 0 & 0 & 0 & 0 & . & . & . \\ C_1 & B_1 & 0 & 0 & 0 & 0 & . & . & . \\ C_{3/2} & B_{3/2} & C_{1/2} & B_{1/2} & 0 & 0 & . & . & . \\ C_2 & B_2 & C_1 & B_1 & 0 & 0 & . & . & . \\ C_{5/2} & B_{5/2} & C_{3/2} & B_{3/2} & C_{1/2} & B_{1/2} & . & . & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix} \begin{Bmatrix} M_1 \\ N_1 \\ M_2 \\ N_2 \\ M_3 \\ . \end{Bmatrix} \quad (10a)$$

or

$$\{R\} \approx [C] \{M\} \quad (10b)$$

where the M,N column is calculated by means of equation (9c) so that

$$\{R\} \approx [C] [D_2] \{G\} \quad (10c)$$

which may also be written as

$$\{R\} \approx [Q_2] \{G\} \quad (10d)$$

where the Q_2 matrix in effect performs the operations of Duhamel's integral.

If values of R are desired at only integral multiples of Δs , alternate rows in equation (10a) may be omitted so that the equation is simplified to

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ . \end{Bmatrix} \approx \begin{bmatrix} C_1 & B_1 & 0 & 0 & 0 & 0 & . & . \\ C_2 & B_2 & C_1 & B_1 & 0 & 0 & . & . \\ C_3 & B_3 & C_2 & B_2 & C_1 & B_1 & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \begin{Bmatrix} M_1 \\ N_1 \\ M_2 \\ N_2 \end{Bmatrix} \quad (10e)$$

Methods for calculating the responses $B(s)$ and $C(s)$ in terms of the indicial response $A(s)$ are presented in the appendix. The steps required in computing responses by the method of parabolic-arc approximation to the forcing function may then be summarized as follows:

1. The values of the forcing function as well as those of the indicial response $A(s)$ are tabulated at the end points and midpoints of the intervals; there will then be $2n$ values of G (from $G_{1/2}$ to G_n) as well as $2n + 1$ values of A (from A_0 to A_n).
2. If values of the response are desired only at the end points of the intervals, the functions B and C can be calculated at those points from equations (A6) and (A14) (disregarding alternate rows). If, on the other hand, values of the response are also desired at the midpoints, the functions B and C must be calculated at those points as well; equations (A7) and (A14) may be used for this purpose. Equations (A8) and (A15) may be used to yield somewhat more accurate values of $B_{1/2}$ and $C_{1/2}$ than furnished by equations (A7) and (A14), respectively.
3. The values of B and C are tabulated in matrices as indicated in equation (10e) or equation (10a), depending on whether the response is to be calculated only at the end points or at both the end points and the midpoints.
4. The values of the coefficients M and N are calculated from the values of G by means of equation (9).
5. The response to the function G is calculated from equations (10a) or (10e), provided G_0 is zero. If G_0 is not zero, a term ($G_0A_1, G_0A_2, \dots$) has to be added to each value of the response calculated from equation (10a) or (10e); the response at $s = 0$ is then G_0A_0 .

ANALYSIS OF THE INVERSE PROBLEMS

There are two types of inverse problems. The one concerned with calculating the forcing function from a known response and indicial response appears to be generally of more interest. It is consequently treated in more detail in this paper than the second inverse problem, which consists of calculating indicial responses from known forcing functions and responses.

Calculation of the Forcing Function

For the purpose of calculating responses to arbitrary forcing functions, the numerical methods presented (equations (6c), (7c), and (10d)) may be considered to consist of summation formulas involving the indicial response, the arbitrary forcing function, as well as certain numerical factors. For the purpose of solving the inverse problem, however, the same equations may be considered to be linear simultaneous equations with known responses on one side and unknown values of the forcing function multiplied by certain coefficients on the other side. These coefficients are the elements of the Q matrices. The unknown values of the forcing function may then be obtained by solving the simultaneous equations in any convenient manner. There must, of course, be as many equations as unknowns, or, in other words, the Q matrices must be square if they are to be used for the solution of the inverse problem.

In the following analysis G_0 is assumed to be zero. When it is not zero it may be calculated from the relation

$$G_0 = \frac{R_0}{A_0} \quad (11)$$

and the terms $G_0 A_1, G_0 A_2, \dots$ must be subtracted from the known values of the response before they are operated upon in the manner indicated in the following sections in order to calculate the forcing function. To the values of the forcing function calculated in this manner the value G_0 as calculated from equation (11) must then be added at each point.

Numerical-integration method.— The numerical evaluation of Duhamel's integral leads to equation (6c). In the case of the inverse problem, a unique solution for the values of G corresponding to a given set of values of R can be obtained from this equation only if A_0 is zero, since otherwise there are more unknown values of G than there are

equations. Consequently, this method can be applied to the analysis of the inverse problem only if A_0 is zero. In that special case the Q matrix is triangular; the solution of equation (6c) can be performed by the following method: The Q matrix may be written in the form:

$$[Q] = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \dots \\ a_{21} & a_{22} & 0 & 0 & \dots \\ a_{31} & a_{32} & a_{33} & 0 & \dots \\ a_{41} & a_{42} & a_{43} & a_{44} & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix} \quad (12)$$

where the coefficients a are obtained by postmultiplying the A matrix of equation (6) by the D matrix of equation (4). The solution of equation (6c) is then

$$\left. \begin{aligned} G_{1/2} &= \frac{1}{a_{11}} R_1 \\ G_{3/2} &= \frac{1}{a_{22}} (R_2 - a_{21} G_{1/2}) \\ G_{5/2} &= \frac{1}{a_{33}} (R_3 - a_{31} G_{1/2} - a_{32} G_{3/2}) \\ \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \right\} \quad (13)$$

If many response functions are to be analyzed for the same indicial response, it may be expedient to invert the Q matrix as follows:

$$[Q]^{-1} = \begin{bmatrix} b_{11} & 0 & 0 & 0 & \dots \\ b_{21} & b_{22} & 0 & 0 & \dots \\ b_{31} & b_{32} & b_{33} & 0 & \dots \\ b_{41} & b_{42} & b_{43} & b_{44} & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix} \quad (14)$$

where each column of $[Q]^{-1}$ is obtained successively by means of equation (13) from an appropriate R column of the type

$$\{R\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{Bmatrix}, \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{Bmatrix}, \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{Bmatrix}, \dots$$

and the values of G pertinent to the given problem.

The steps required in calculating forcing functions by this method are:

1. The n values of the response are tabulated at the end points of the intervals from points 1 to n .
2. The A matrix is obtained from equation (6) as described in the section concerned with the direct problem; it is postmultiplied by the D matrix of equation (4) to yield the Q matrix.
3. The elements of the Q matrix are considered to be the coefficients of simultaneous equations for the unknown values of G from $G_1/2$ to $G_{n-1/2}$ in terms of the known values from R_1 to R_n . The solution of these equations may be carried out as outlined in equation (13).
4. If it is desired to invert the Q matrix, equation (14) can be used. The values of the forcing function corresponding to any set of values of the response may then be obtained by premultiplying the values of the response by the inverse of the Q matrix.

Straight-line approximation.— The straight-line approximation to the forcing function leads to equation (7c), which involves the triangular matrix

$$[Q_1] = \begin{bmatrix} a_1 & 0 & 0 & 0 & \dots \\ a_2 & a_1 & 0 & 0 & \dots \\ a_3 & a_2 & a_1 & 0 & \dots \\ a_4 & a_3 & a_2 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (15)$$

where

$$\left. \begin{aligned} a_1 &= B_1 \\ a_2 &= B_2 - B_1 \\ a_3 &= B_3 - B_2 \\ a_4 &= B_4 - B_3 \\ &\dots \end{aligned} \right\} \quad (16)$$

When there are n values of G (from G_1 to G_n) and n values of R (from R_1 to R_n), the Q_1 matrix is square and there are n equations for the n unknown values of G .

The solution for these values is carried out very easily as a result of the triangular form of the Q_1 matrix. In fact

$$\left. \begin{aligned} G_1 &= \frac{1}{a_1} R_1 \\ G_2 &= \frac{1}{a_1} (R_2 - a_2 G_1) \\ G_3 &= \frac{1}{a_1} (R_3 - a_3 G_1 - a_2 G_2) \\ G_4 &= \frac{1}{a_1} (R_4 - a_4 G_1 - a_3 G_2 - a_2 G_3) \\ &\dots \end{aligned} \right\} \quad (17)$$

If the forcing functions for many known responses are to be obtained for the same system (with the same indicial response), it may be more expedient to calculate the inverse of the Q_1 matrix than to solve for each forcing function separately. The inverse of the Q_1 matrix may be written as

$$[Q_1]^{-1} = \begin{bmatrix} b_1 & 0 & 0 & 0 & \dots \\ b_2 & b_1 & 0 & 0 & \dots \\ b_3 & b_2 & b_1 & 0 & \dots \\ b_4 & b_3 & b_2 & b_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (18)$$

where

$$\left. \begin{aligned} b_1 &= \frac{1}{a_1} \\ b_2 &= -\frac{1}{a_1} a_2 b_1 \\ b_3 &= -\frac{1}{a_1} (a_3 b_1 - a_2 b_2) \\ b_4 &= -\frac{1}{a_1} (a_4 b_1 - a_3 b_2 - a_2 b_3) \\ &\vdots \end{aligned} \right\} \quad (19)$$

The steps required in calculating forcing functions by this method are:

1. The values of the response are tabulated at the end points of the intervals from points 1 to n.

2. The Q_1 matrix is calculated as indicated for the direct problem.

3. A set of simultaneous equations having the elements of the Q_1 matrix as coefficients and the values of R as knowns is solved for the unknown values of the forcing function G_1 to G_n as indicated in equation (17).

4. If it is desired to invert the Q_1 matrix, equations (18) and (19) can be used; the forcing functions may then be obtained by premultiplying the given sets of values of R by the inverse of the Q_1 matrix.

Parabolic-arc approximation.— The Q_2 matrix obtained for the parabolic-arc method (equation (10d)) is not quite triangular but may be reduced to triangular form by partitioning:

$$\begin{aligned}
 [Q_2] &= \begin{bmatrix} c_1 & d_1 & 0 & 0 & 0 & 0 & \dots \\ e_1 & f_1 & 0 & 0 & 0 & 0 & \dots \\ \hline c_2 & d_2 & c_1 & d_1 & 0 & 0 & \dots \\ e_2 & f_2 & e_1 & f_1 & 0 & 0 & \dots \\ \hline c_3 & d_3 & c_2 & d_2 & c_1 & d_1 & \dots \\ e_3 & f_3 & e_2 & f_2 & e_1 & f_1 & \dots \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix} \\
 &= \begin{bmatrix} \underline{g_1} & \underline{0} & \underline{0} & \dots \\ \underline{g_2} & \underline{g_1} & \underline{0} & \dots \\ \underline{g_3} & \underline{g_2} & \underline{g_1} & \dots \\ \cdot & \cdot & \cdot & \dots \end{bmatrix} \tag{20}
 \end{aligned}$$

where

$$\left. \begin{aligned} \underline{g_1} &\equiv \begin{bmatrix} c_1 & d_1 \\ e_1 & f_1 \end{bmatrix} \\ \underline{g_2} &\equiv \begin{bmatrix} c_2 & d_2 \\ e_2 & f_2 \end{bmatrix} \\ \underline{0} &\equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \right\} \tag{20a}$$

and where the c , d , e , and f values are obtained by postmultiplying the C matrix of equation (10b) by the D_2 matrix of equation (9d) to yield the Q_2 matrix. If the column matrix of response values is also partitioned into pairs such that

$$\{R\} = \begin{Bmatrix} R_1/2 \\ R_1 \\ R_3/2 \\ R_2 \\ . \end{Bmatrix} = \begin{Bmatrix} \underline{R}_1 \\ \underline{R}_2 \\ . \end{Bmatrix} \quad (21)$$

where

$$\left. \begin{aligned} \underline{R}_1 &\equiv \begin{Bmatrix} R_1/2 \\ R_1 \end{Bmatrix} \\ \underline{R}_2 &\equiv \begin{Bmatrix} R_3/2 \\ R_2 \end{Bmatrix} \end{aligned} \right\} \quad (21a)$$

then a solution of the simultaneous equations having the elements of the Q_2 matrix as coefficients may be effected in the manner indicated for the straight-line method, except that two-by-two matrices take the place of ordinary algebraic quantities. The $\underline{g}$ matrices of equations (20a) take the place of the coefficients a (equations (15) and (16)), the $\underline{R}$ matrices of equations (21a) take the place of the values of R in equation (17), and the forcing function is computed in the form of $\underline{G}$ matrices defined by

$$\left. \begin{aligned} \underline{G}_1 &\equiv \begin{Bmatrix} G_1/2 \\ G_1 \end{Bmatrix} \\ \underline{G}_2 &\equiv \begin{Bmatrix} G_3/2 \\ G_2 \end{Bmatrix} \end{aligned} \right\} \quad (22)$$

which take the place of the values of G in equation (17). If the Q_2 matrix is to be inverted, the resulting matrix has the form

$$[Q_2]^{-1} = \begin{bmatrix} \underline{h}_1 & 0 & 0 & \dots \\ \underline{h}_2 & \underline{h}_1 & 0 & \dots \\ \underline{h}_3 & \underline{h}_2 & \underline{h}_1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (23)$$

where the $\underline{h}$ matrices take the place of values of b in equation (18) and are calculated as indicated in equation (19). The quantity $1/a_1$ is replaced by $\underline{g}_1^{-1}$, the inverse of the $\underline{g}_1$ matrix. The operations called for in equations (17) and (19) are now, of course, matrix multiplications, so that the order of the multiplicands must be preserved. The fact that the inversion of the Q_2 matrix entails, with the exception of the inversion of the two-by-two $\underline{g}_1$ matrix, only matrix multiplications and additions of low-order matrices facilitates the problem greatly.

The steps required to calculate forcing functions by this method are:

1. The values of the response are tabulated at the end points and midpoints of the intervals from $R_1/2$ to R_n .
2. The values of B and C are calculated at both the end points and the midpoints of the intervals and entered in a matrix as shown in equation (10a). This matrix is postmultiplied by the D_2 matrix shown in equation (9c) to yield the Q_2 matrix.
3. A set of simultaneous equations having the elements of the Q_2 matrix as coefficients and the values of R as knowns is solved for the unknown values of the forcing function $G_1/2$ to G_n in the manner suggested in equation (17), where the submatrices indicated in equations (20a), (21a), and (22) are used instead of the terms a , R , and G of equation (17).
4. If it is preferred to solve the problem by inverting the Q_2 matrix, the inversion may be performed as indicated in equations (18) and (19), where the submatrices indicated in equation (20a) are used instead of the values of a of equation (19), however; the forcing functions may then be obtained by premultiplying the given sets of values of R by the inverse of the Q_2 matrix.

Calculation of the Indicial Response

The numerical-integration method outlined for the direct problem may in certain cases be modified to obtain a solution to the second inverse problem of calculating an indicial response from a known forcing function and response. Equation (6) may be rewritten in the form:

$$\begin{bmatrix} \frac{1}{2}G'_1 & \frac{1}{2}G'_0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3}G'_2 & \frac{4}{3}G'_1 & \frac{1}{3}G'_0 & 0 & 0 & 0 & \dots \\ \frac{5}{12}G'_3 & G'_2 & \frac{5}{4}G'_1 & \frac{1}{3}G'_0 & 0 & 0 & \dots \\ \frac{2}{3}G'_4 & \frac{4}{3}G'_3 & \frac{2}{3}G'_2 & \frac{4}{3}G'_1 & \frac{1}{3}G'_0 & 0 & \dots \\ \frac{5}{12}G'_5 & G'_4 & \frac{5}{4}G'_3 & \frac{2}{3}G'_2 & \frac{4}{3}G'_1 & \frac{1}{3}G'_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ \vdots \end{Bmatrix} \approx \frac{1}{\Delta s} \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ \vdots \end{Bmatrix} \quad (24)$$

which permits a solution for the values of A from known values of R and G' . The values of G' may be obtained from the values of G at the midpoint of the chosen intervals by means of equation (4). As in equation (6), it is assumed that G_0 is zero. Furthermore unless either A_0 or G_0 is zero as well, equation (24) will have more unknown values of A than equations, so that it cannot be solved.

If these conditions are not met, it may be possible to use another approach which consists of using the alternate form of Duhamel's integral:

$$R(s) = \int_0^s A'(\sigma)G(s - \sigma)d\sigma \quad (25)$$

The following equations may be obtained in a manner analogous to that used in obtaining equation (6):

$$\begin{bmatrix} \frac{1}{2}G_1 & \frac{1}{2}G_0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3}G_2 & \frac{4}{3}G_1 & \frac{1}{3}G_0 & 0 & 0 & 0 & \dots \\ \frac{1}{3}G_3 & \frac{5}{4}G_2 & G_1 & \frac{5}{12}G_0 & 0 & 0 & \dots \\ \frac{1}{3}G_4 & \frac{4}{3}G_3 & \frac{2}{3}G_2 & \frac{4}{3}G_1 & \frac{1}{3}G_0 & 0 & \dots \\ \frac{1}{3}G_5 & \frac{4}{3}G_4 & \frac{2}{3}G_3 & \frac{5}{4}G_2 & G_1 & \frac{5}{12}G_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} A'_0 \\ A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \\ \vdots \end{Bmatrix} \approx \frac{1}{\Delta s} \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ \vdots \end{Bmatrix} \quad (26)$$

This equation serves to solve for the values of A' in terms of the known values of the response; the values of the forcing function appear as coefficients in the simultaneous equations, as do the integrating factors. Equation (25) is valid when A_0 is zero; it does not imply any restriction on G_0 . However, unless either G_0 or A_0 is zero there will be more unknown values of A than equations in equation (26), so that the equation cannot be solved. A set of values of A may be obtained from the calculated values of A' by means of an integrating matrix:

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ . \end{Bmatrix} = \Delta s \begin{bmatrix} 5/12 & 2/3 & -1/12 & 0 & 0 & 0 & . & . & . \\ 1/3 & 4/3 & 1/3 & 0 & 0 & 0 & . & . & . \\ 5/12 & 1 & 5/4 & 1/3 & 0 & 0 & . & . & . \\ 1/3 & 4/3 & 2/3 & 4/3 & 2/3 & 0 & . & . & . \\ 5/12 & 1 & 5/4 & 2/3 & 4/3 & 1/3 & . & . & . \\ . & . & . & . & . & . & . & . & . \end{bmatrix} \begin{Bmatrix} A'_0 \\ A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \\ . \end{Bmatrix} \quad (27)$$

The integrating process indicated in equation (27) will tend to average out any discrepancies in the values of A' calculated from equation (26).

Since in the particular cases where equations (24) or (26) may be used the matrices of the values of G or G' will be triangular, the methods indicated in the preceding sections in analyzing the numerical integration method will also be applicable to the solution of these equations or the inversion of these matrices.

RESULTS AND DISCUSSION

Comparison of Numerical and Exact Results

The direct problem.— The accuracy of the numerical results has been investigated for several cases by comparing their results with known exact solutions for simple indicial response and forcing functions. The indicial response selected is:

$$A(s) = 1 - \frac{1}{2} e^{-0.2s} \quad (28)$$

Two forcing functions have been considered

$$G_I(s) = \sin \frac{\pi}{24} s \quad (29)$$

and

$$G_{II}(s) = \frac{1}{2} \left(1 - \cos \frac{\pi}{24} s \right) \quad (30)$$

The responses to the two forcing functions have been computed by Duhamel's integral exactly and by the numerical methods of this paper. The interval was chosen as $\Delta s = 2$, which is one twenty-fourth of the period of the forcing function. The response $B(s)$ to the unit gradient and the response $C(s)$ to the unit parabolic function required in the second method (the straight-line or parabolic-arc approximations) have been computed by the numerical methods indicated by equations (A7) and (A14); they are presented in table I with the exact values. A comparison shows that the numerical methods for the evaluation of the unit responses $B(s)$ and $C(s)$ should yield results to an accuracy of better than 0.1 percent for reasonably small intervals.

The response to the forcing function of equation (29) has been calculated by all three numerical methods and is presented in table II.

A comparison of the results indicates that the numerical-integration method and the straight-line approximation have the same accuracy, approximately 0.2 percent. The parabolic-arc approximation gives much better accuracy, approximately 0.01 percent in this example.

The inverse problem.— The forcing functions which give rise to the responses may be calculated from these exact responses and the indicial response given by equation (29). Since A_0 is not zero, the approximate-integration method cannot be used. Therefore, only the other two methods have been used to calculate the forcing functions.

An increment $\Delta s = 2$ was used and the required unit gradient and unit parabolic responses were taken from table I. The forcing functions calculated by these methods should check those given exactly by equations (30) and (31). The comparison is presented in table III. The average accuracy of the straight-line approximation applied to the inverse problem is approximately 0.3 percent; that of the parabolic-arc approximation, approximately 0.04 percent. No points have been calculated at the midpoints of the intervals by means of the straight-line

approximation since they would simply be the average of the ordinates at the end points of the given intervals in view of the approximation inherent in that method.

Factors Affecting the Accuracy of the Methods

The comparisons of the numerical methods with known solutions show that, with intervals of the order of $1/20$ to $1/30$ of the period of the first natural frequency of the system or that of the forcing function, the numerical-integration method and the straight-line approximation yield results that have an accuracy of 1 percent or better; whereas the results of the parabolic-arc approximation are accurate to 0.1 percent or better. Similar calculations have indicated that an accuracy comparable to these values may be expected even when higher harmonics are present, provided those higher than the third are unimportant compared with the first few.

In general, the accuracy of the numerical methods depends to some extent on the shape of the curve that is approximated; the closer the approximate curve (which consists of straight-line or parabolic-arc segments) fits the actual curve, the higher the accuracy. Consequently, the interval should be chosen small enough that the degree of curvature of the approximated curve is small within a segment; any reflexes in the approximated curve should be near end points of segments, if possible.

In the case of the inverse problem, great care must be exercised in the case where both the indicial response and its first derivative vanish initially ($A_0 = A'_0 = 0$). The determinant of the coefficients of the simultaneous equations to be solved for the values of the forcing function tends to be relatively small for this case and the equations relatively ill-behaved. It is then difficult to obtain reliable results by numerical means unless very small increments are taken near $s = 0$. The accuracy of the values of the forcing function may, however, be improved by replacing the actual indicial response by a straight-line segment extending over the first two or three intervals for the purpose of computation; in this manner a nonvanishing value is assigned to A'_0 and the simultaneous equations for the values of G can be solved more accurately.

Factors Affecting Choice of Methods

The accuracy of the numerical-integration method and straight-line approximation for both problems tends to be approximately the same. The numerical-integration method is less time consuming than the other method if only one response or one forcing function is to be analyzed, since the

straight-line approximation requires the computation of the function B . If many cases are to be analyzed for the same indicial response, the straight-line approximation will tend to be the more expedient, since it involves matrices which are somewhat more convenient to use. The numerical-approximation method is, of course, applicable to the inverse problem only when A_0 is zero.

The parabolic-arc approximation is more accurate than the other methods but is also more time consuming. If by decreasing the size of the segments used in the other methods their accuracy is increased until it is comparable to that of the parabolic-arc approximation, the expenditure of time required becomes comparable as well. The shape of the function to be approximated will then determine whether any small advantage would be with the parabolic-arc approximation or the other methods.

CONCLUDING REMARKS

Two matrix methods for solving the problem of calculating forcing functions from known responses have been derived by means of a numerical analysis of the problem of calculating responses from forcing functions. The first method, which consists of an approximate evaluation of Duhamel's integral, leads to a matrix that can only be inverted when the initial value of the indicial response is zero. The matrices obtained in the other method, which consists of approximating the forcing function by straight-line or parabolic-arc segments, can always be inverted readily. Some results obtained by the numerical methods for simple explicit functions have been compared with exact results. The accuracy of the numerical methods was found to be adequate for most practical purposes. A brief discussion has been given of the possibility of calculating the indicial response by the approximate-integration method if the forcing function and the response to it are known.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., July 14, 1949

APPENDIX

EVALUATION OF RESPONSES TO UNIT GRADIENT
AND UNIT PARABOLIC FORCING FUNCTIONS

The response to the unit gradient function can be determined from the indicial response by means of Duhamel's integral. For the unit gradient, for $0 \leq s \leq \Delta s$,

$$G(s) = \frac{s}{\Delta s}$$

and

$$G'(s) = \frac{1}{\Delta s}$$

for $s \geq \Delta s$,

$$G(s) = 1$$

and

$$G'(s) = 0$$

so that, for $0 \leq s \leq \Delta s$,

$$B(s) = \frac{1}{\Delta s} \int_0^s A(s - \sigma) d\sigma \quad (A1)$$

and, for $s \geq \Delta s$,

$$B(s) = \frac{1}{\Delta s} \int_0^{\Delta s} A(s - \sigma) d\sigma \quad (A2)$$

or, if ξ is substituted for $s - \sigma$ and ξ is substituted for $-(\sigma - \Delta s)$, for $0 \leq s \leq \Delta s$,

$$B(s) = \frac{1}{\Delta s} \int_0^s A(\xi) d\xi \quad (A3)$$

and, for $s_n \geq \Delta s$,

$$B_n = \frac{1}{\Delta s} \int_0^{\Delta s} A(s_{n-1} + \xi) d\xi \quad (A4)$$

If equations (A3) and (A4) cannot be evaluated in closed form, numerical methods can be used. Since the accuracy of the further calculations (equation (7a)) depends on the accuracy with which the function B has been calculated, increments $\Delta s/2$ should be used in performing the integration indicated by equations (A3) and (A4). However, the values of B used need only be calculated at integral multiples of Δs . If Simpson's integration rule is used to evaluate equation (A4), the following values are obtained for B :

$$\left. \begin{aligned} B_1 &\approx \frac{\Delta s/2}{\Delta s} \frac{1}{3} (A_0 + 4A_{1/2} + A_1) \\ B_2 &\approx \frac{\Delta s/2}{\Delta s} \frac{1}{3} (A_1 + 4A_{3/2} + A_2) \\ &\vdots \end{aligned} \right\} \quad (A5)$$

or

$$\begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \end{Bmatrix} \approx \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{Bmatrix} A_0 \\ A_{1/2} \\ A_1 \\ A_{3/2} \\ \vdots \end{Bmatrix} \quad (A6)$$

If for some reason the values of the function B are needed at the midpoints of the intervals, they can be calculated in a similar manner. The value at $\Delta s/2$ poses a special problem in that the Simpson factors are not directly applicable. However, the integration of function A

between 0 and $\Delta s/2$ may be carried out approximately by replacing A by a parabolic segment between 0 and Δs but integrating the parabola only between 0 and $\Delta s/2$. This procedure leads to the factors $5/12$, $2/3$, and $-1/12$ which are used instead of the Simpson factors $1/3$, $4/3$, and $1/3$. Thus,

$$\begin{Bmatrix} B_{1/2} \\ B_1 \\ B_{3/2} \\ B_2 \\ B_{5/2} \\ \cdot \end{Bmatrix} \approx \frac{1}{6} \begin{bmatrix} 5/4 & 2 & -1/4 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} A_0 \\ A_{1/2} \\ A_1 \\ A_{3/2} \\ A_2 \\ \cdot \end{Bmatrix} \quad (A7)$$

Should a more accurate value of $B_{1/2}$ be desired, it can be obtained from

$$\begin{aligned} B_{1/2} &\approx \frac{\Delta s/4}{\Delta s} \frac{1}{3} (A_0 + 4A_{1/4} + A_{1/2}) \\ &\approx \frac{1}{12} (A_0 + 4A_{1/4} + A_{1/2}) \end{aligned} \quad (A8)$$

The response to the forcing function as approximated by equation (8) is composed of two parts. One part is due to a unit gradient $B(s)$ which has been described in the preceding paragraphs. The second part of the response, that due to the unit parabolic function $C(s)$, may similarly be calculated from the indicial response. For the parabolic function, for $0 \leq s \leq \Delta s$,

$$G(s) = \left(\frac{s}{\Delta s} \right)^2$$

and

$$G'(s) = \frac{2s}{(\Delta s)^2}$$

for $s \geq \Delta s$,

$$G(s) = 1$$

and

$$G^*(s) = 0$$

so that, for $0 \leq s \leq \Delta s$,

$$C(s) = \frac{2}{(\Delta s)^2} \int_0^s A(s - \sigma) \sigma \, d\sigma \quad (A9)$$

for $s \geq \Delta s$,

$$C(s) = \frac{2}{(\Delta s)^2} \int_0^{\Delta s} A(s - \sigma) \sigma \, d\sigma \quad (A10)$$

or, if ξ is substituted for $s - \sigma$ and ξ is substituted for $-(\sigma - \Delta s)$, for $0 \leq s \leq \Delta s$,

$$C(s) = \frac{2}{\Delta s} \int_0^s A(\xi) \left(\frac{s}{\Delta s} - \frac{\xi}{\Delta s} \right) d\xi \quad (A11)$$

for $s \geq \Delta s$,

$$C_n = \frac{2}{\Delta s} \int_0^{\Delta s} A(s_{n-1} + \xi) \left(1 - \frac{\xi}{\Delta s} \right) d\xi \quad (A12)$$

If equations (A11) and (A12) cannot be evaluated in closed form, numerical methods may be employed. The factor $1 - \frac{\xi}{\Delta s}$ takes the values 1, 1/2, and 0 when the term $s_n - \sigma$ is s_{n-1} , $s_{n-1} - \frac{1}{2}$, and s_n , respectively, as does the factor $\frac{s}{\Delta s} - \frac{\xi}{\Delta s}$ for $s = \Delta s$. However, for $s = \frac{\Delta s}{2}$ the factor $\frac{s}{\Delta s} - \frac{\xi}{\Delta s}$ assumes the values 1/2, 0, and -1/2 when σ is $\Delta s/2$, 0, and $-\Delta s/2$; the point at $\sigma = -\frac{\Delta s}{2}$ is fictitious since it lies beyond the limits of the integration and is used only to furnish a better description of the curve in the region of interest.

These values may be combined with the integrating factors to yield

$$\left. \begin{aligned} c_{1/2} &\approx \frac{2}{\Delta s} \frac{\Delta s}{2} \left(\frac{5}{12} \frac{1}{2} A_0 + \frac{2}{3} (0) A_{1/2} - \frac{1}{12} \left(-\frac{1}{2} \right) A_1 \right) \\ c_1 &\approx \frac{2}{\Delta s} \frac{\Delta s}{2} \frac{1}{3} \left(1 A_0 + \frac{4}{2} A_{1/2} + (0) A_1 \right) \\ c_{3/2} &\approx \frac{2}{\Delta s} \frac{\Delta s}{2} \frac{1}{3} \left(1 A_{1/2} + \frac{4}{2} A_1 + (0) A_{3/2} \right) \\ c_2 &\approx \frac{2}{\Delta s} \frac{\Delta s}{2} \frac{1}{3} \left(1 A_1 + \frac{4}{2} A_{3/2} + (0) A_2 \right) \end{aligned} \right\} \quad (A13)$$

or, in matrix form,

$$\begin{bmatrix} c_{1/2} \\ c_1 \\ c_{3/2} \\ c_2 \\ c_{5/2} \\ . \end{bmatrix} \approx \frac{1}{3} \begin{bmatrix} 5/8 & 0 & 1/8 & 0 & 0 & . & . & . \\ 1 & 2 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 2 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 2 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & 2 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} A_0 \\ A_{1/2} \\ A_1 \\ A_{3/2} \\ A_2 \\ . \end{bmatrix} \quad (A14)$$

Again, as in equation (A6), the rows in equation (A14) for the values of C at the midpoints of the intervals may be disregarded if these values are not of interest.

A slightly more accurate value of $c_{1/2}$ than that given in equations (A13) and (A14) is given by

$$\begin{aligned} c_{1/2} &\approx \frac{2}{\Delta s} \frac{\Delta s}{4} \frac{1}{3} \left(\frac{1}{2} A_0 + \frac{4}{4} A_{1/4} + (0) A_{1/2} \right) \\ &\approx \frac{1}{12} (A_0 + 2A_{1/4}) \end{aligned} \quad (A15)$$

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TABLE I.— COMPARISON OF THE NUMERICAL AND EXACT
CALCULATIONS OF THE RESPONSES TO THE UNIT
GRADIENT AND UNIT PARABOLIC FUNCTIONS

s	Response to unit gradient function, B(s)		Response to unit parabolic function, C(s)	
	Exact	Numerical	Exact	Numerical
0	0	0	0	0
1	.5462	.5468	.5317	.5317
2	.6290	.6290	.6166	.6166
3	.6962	.6963	.6861	.6861
4	.7512	.7513	.7430	.7430
5	.7965	.7964	.7896	.7896
6	.8332	.8333	.8277	.8277
7	.8635	.8636	.8589	.8590
8	.8882	.8882	.8845	.8845
9	.9085	.9079	.9054	.9048
10	.9250	.9258	.9226	.9233
11	.9388	.9396	.9366	.9367
12	.9497	.9497	.9481	.9481
13	.9589	.9589	.9575	.9575
14	.9664	.9664	.9652	.9653
15	.9724	.9724	.9715	.9715
16	.9774	.9774	.9767	.9767
17	.9816	.9815	.9809	.9809
18	.9848	.9849	.9844	.9844
19	.9876	.9876	.9872	.9872
20	.9899	.9899	.9895	.9895


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TABLE II.-- COMPARISON OF EXACT AND APPROXIMATE
VALUES OF THE RESPONSE TO THE FUNCTION

$$G(s) = \sin \frac{\pi}{24}s$$

Method s	Exact	Approximate integration, equations (4) and (6)	Straight line, equation (7a)	Parabolic segment, equations (9c) and (10a)
0	0	0	0	0
2	.1522	.1498	.1522	.1522
4	.3294	.3287	.3292	.3293
6	.5079	.5072	.5073	.5079
8	.6677	.6661	.6671	.6678
10	.7927	.7914	.7912	.7927
12	.8708	.8689	.8689	.8707
14	.8944	.8926	.8921	.8944
16	.8600	.8577	.8581	.8600
18	.7693	.7679	.7671	.7693
20	.6276	.6259	.6255	.6276
22	.4441	.4432	.4423	.4441
24	.2310	.2303	.2296	.2310
26	.0026	.0025	.0017	.0026
28	-.2257	-.2252	-.2260	-.2257
30	-.4385	-.4378	-.4386	-.4385
32	-.6212	-.6196	-.6207	-.6212
34	-.7615	-.7602	-.7602	-.7615
36	-.8499	-.8476	-.8482	-.8499
38	-.8803	-.8787	-.8782	-.8804
40	-.8506	-.8484	-.8488	-.8506
42	-.7629	-.7616	-.7639	-.7629
44	-.6233	-.6226	-.6214	-.6233
46	-.4411	-.4404	-.4395	-.4411
48	-.2289	-.2284	-.2278	-.2290

TABLE III.— COMPARISON OF THE RESULTS OF THE NUMERICAL CALCULATIONS
OF THE FORCING FUNCTIONS WITH THE EXACT VALUES

Forcing function s	$\sin \frac{\pi s}{24}$			$\frac{1}{2} \left(1 - \cos \frac{\pi s}{24} \right)$		
	Exact value	Approximation		Exact value	Approximation	
		Straight- line	Parabolic- arc		Straight- line	Parabolic- arc
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	.1305	-----	.1306	.0043	-----	.0043
2	.2588	.2588	.2586	.0170	.0161	.0170
3	.3827	-----	.3823	.0381	-----	.0373
4	.5000	.5005	.4997	.0670	.0660	.0670
5	.6088	-----	.6085	.1032	-----	.1033
6	.7071	.7081	.7068	.1464	.1453	.1466
7	.7934	-----	.7928	.1956	-----	.1957
8	.8660	.8676	.8659	.2500	.2491	.2502
9	.9239	-----	.9237	.3087	-----	.3086
10	.9659	.9681	.9657	.3706	.3698	.3706
11	.9914	-----	.9913	.4348	-----	.4346
12	1.0000	1.0024	.9997	.5000	.4996	.5000
13	.9914	-----	.9914	.5653	-----	.5649
14	.9659	.9687	.9660	.6292	.6292	.6294
15	.9239	-----	.9234	.6914	-----	.6914
16	.8660	.8685	.8659	.7500	.7500	.7498
17	.7934	-----	.7935	.8044	-----	.8042
18	.7071	.7095	.7071	.8536	.8541	.8534
19	.6088	-----	.6089	.8968	-----	.8967
20	.5000	.5021	.5000	.9330	.9337	.9328
21	.3827	-----	.3829	.9620	-----	.9618
22	.2588	.2604	.2588	.9830	.9841	.9830
23	.1305	-----	.1308	.9958	-----	.9955
24	.0000	.0010	.0000	1.0000	1.0012	.9998
25	-.1305	-----	-.1303	.9958	-----	.9957
26	-.2588	-.2584	-.2587	.9930	.9844	.9931
27	-.3827	-----	-.3822	.9618	-----	.9606
28	-.5000	-.5003	-.4998	.9330	.9341	.9321
29	-.6088	-----	-.6083	.8968	-----	.8965
30	-.7071	-.7081	-.7071	.8536	.8548	.8533
31	-.7934	-----	-.7931	.8044	-----	.8042
32	-.8660	-.8676	-.8659	.7500	.7509	.7500
33	-.9239	-----	-.9237	.6914	-----	.6913
34	-.9659	-.9680	-.9657	.6294	.6301	.6293
35	-.9914	-----	-.9911	.5653	-----	.5661
36	-1.0000	-1.0024	-.9998	.5000	.5004	.4996
37	-.9914	-----	-.9912	.4348	-----	.4338
38	-.9659	-.9685	-.9658	.3706	.3708	.3707
39	-.9239	-----	-.9234	.3087	-----	.3086
40	-.8660	-.8685	-.8659	.2500	.2497	.2499
41	-.7934	-----	-.7933	.1956	-----	.1959
42	-.7071	-.7093	-.7069	.1464	.1457	.1463
43	-.6088	-----	-.6087	.1032	-----	.1033
44	-.5000	-.5019	-.4998	.0670	.0661	.0670
45	-.3827	-----	-.3829	.0381	-----	.0382
46	-.2588	-.2601	-.2585	.0170	.0157	.0169
47	-.1305	-----	-.1308	.0043	-----	.0045
48	.0000	-.0008	.0003	.0000	-.0012	.0000

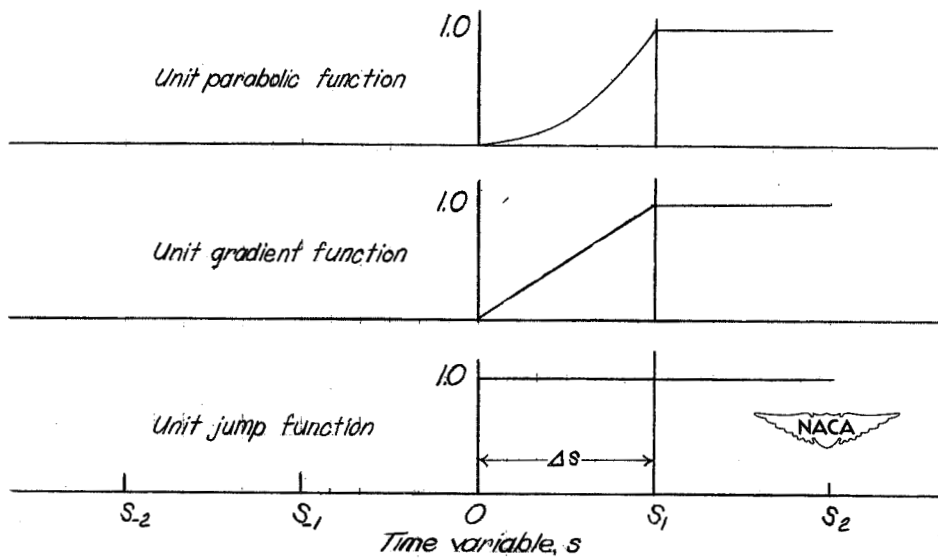


Figure 1. - Unit jump, gradient, and parabolic functions.

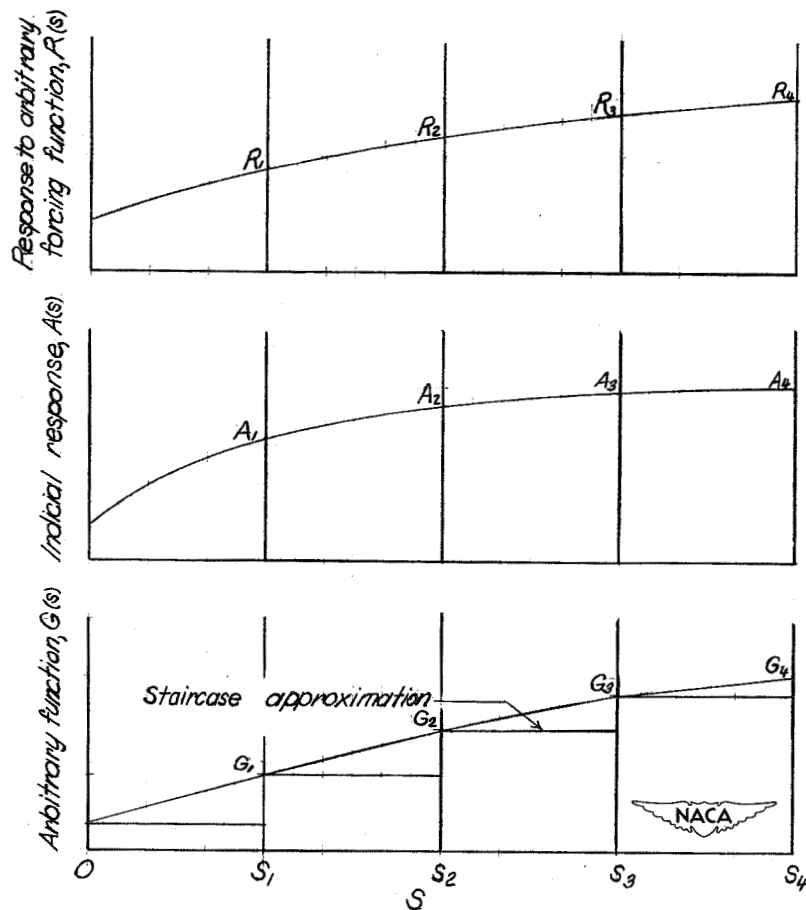


Figure 2. - Arbitrary function, indicial response, and response to arbitrary function.

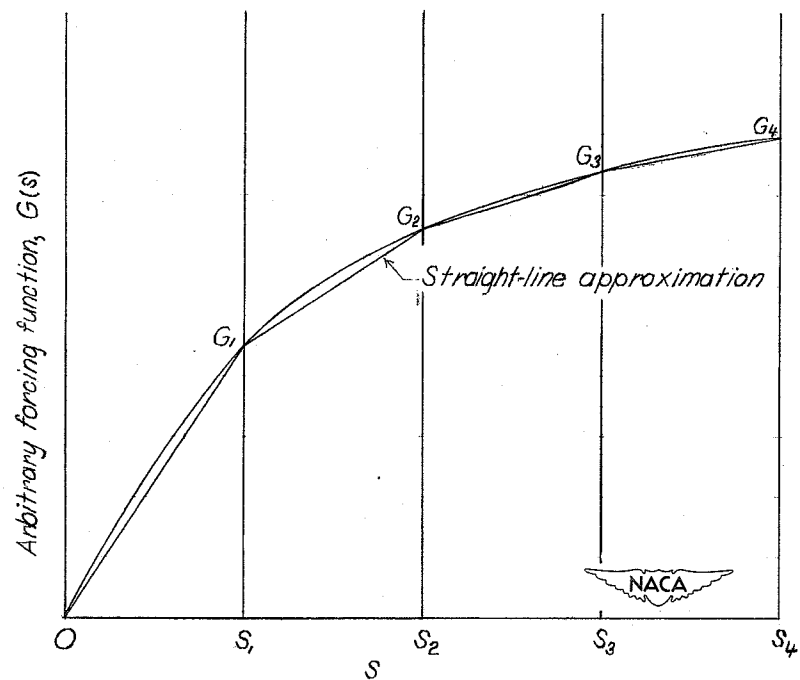


Figure 3.- Straight-line approximation of the forcing function.

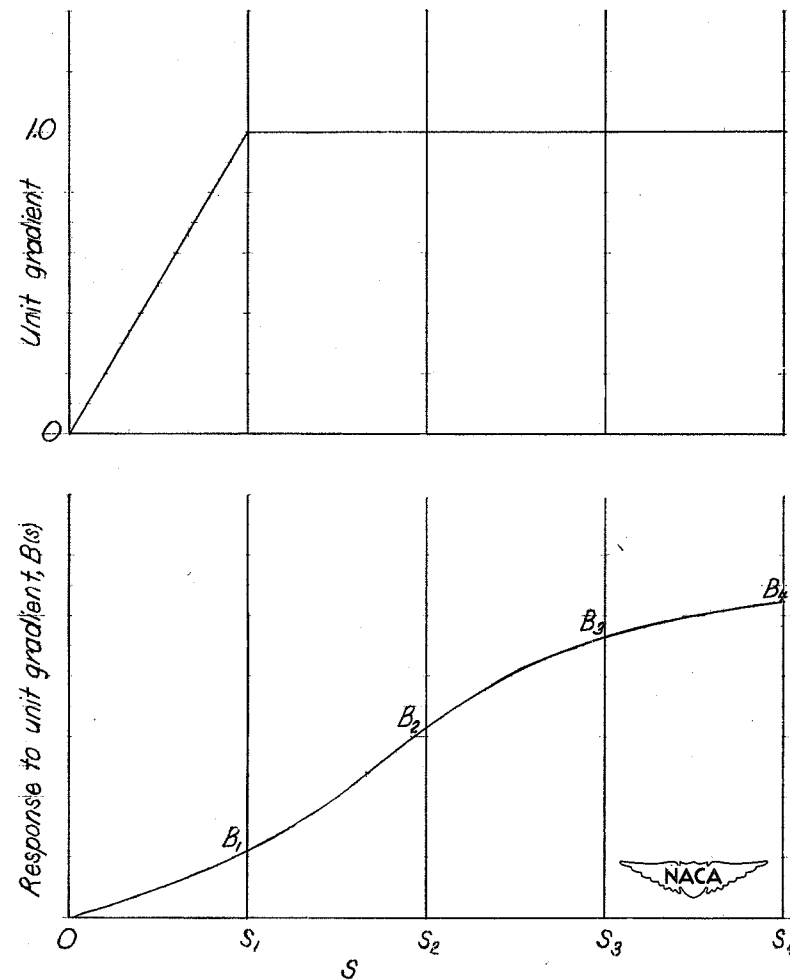


Figure 4.- Response to unit gradient.

Abstract

Two matrix methods for calculating forcing functions from known responses are presented, one consisting of a numerical evaluation of Duhamel's integral and one consisting of straight-line or parabolic-arc approximations to the forcing function. The methods are suitable for application to many dynamic and some static problems.

Abstract

Two matrix methods for calculating forcing functions from known responses are presented, one consisting of a numerical evaluation of Duhamel's integral and one consisting of straight-line or parabolic-arc approximations to the forcing function. The methods are suitable for application to many dynamic and some static problems.